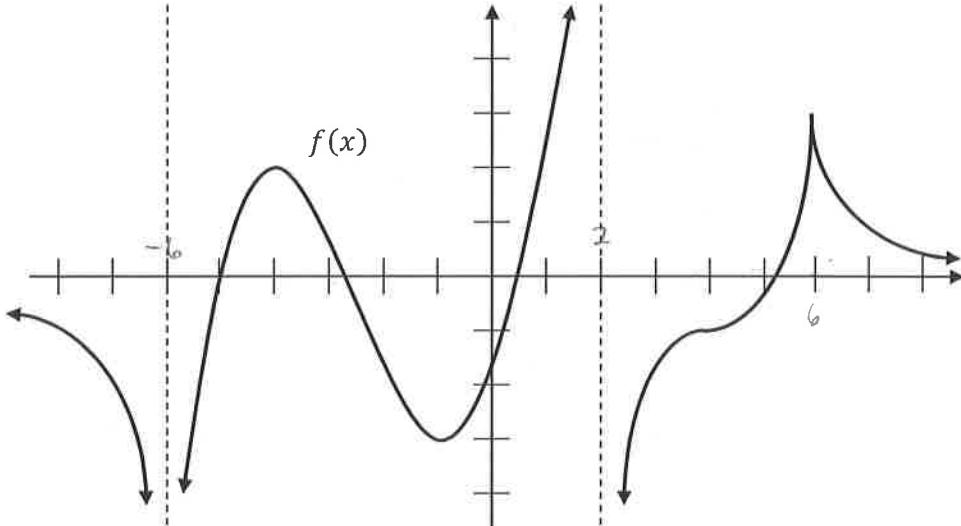


Worksheet 10 – Increasing/Decreasing, Concavity, and Curve Sketching

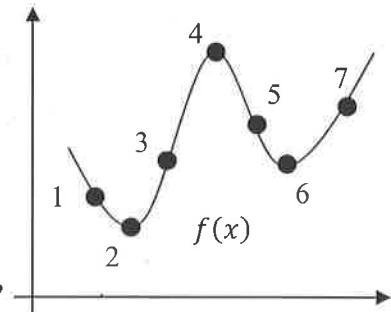
1. Use the graph of $f(x)$ below to answer parts (a) through (h).



- Find the intervals where $f(x)$ is increasing. $(-6, -4] \cup [-1, 2) \cup (2, \infty)$
- Find the intervals where $f(x)$ is decreasing. $(-\infty, -6) \cup [-4, -1] \cup [6, \infty)$
- Find all the x -values where the slope of $f(x)$ is zero. $x = -4, -1, 2$
- Find all the x -values where the derivative of $f(x)$ does not exist. $x = -6, 2, 3$
- Find all the critical points of $f(x)$. $x = -4, -1, 2$
- Find the coordinates of all the relative maxima of $f(x)$. $(-4, 2), (2, 3)$
- Find the coordinates of all the relative minima of $f(x)$. $(-1, -3)$
- Find all the x -values where $f(x)$ changes from increasing or decreasing to increasing or decreasing. $x = -6, -4, -1, 2$

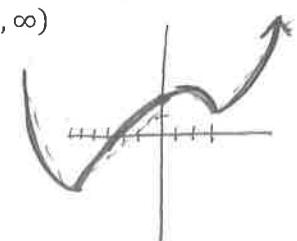
2. Use the numbered points on the graph of $f(x)$ at the right to answer parts (a) through (h).

- At which points is the function increasing? 2, 3, 4, 6, 7
- At which points is there a relative maximum? 4
- At which points is the slope negative? 1, 5
- At which points is the slope zero? 2, 4, 6
- At which points is there a relative minimum? 2, 6
- At which points is the function decreasing? 1, 2, 4, 5, 6
- At which points is the slope positive? 3, 7



3. Sketch the graph of a single continuous function with a y -intercept at $(0, 2)$ that has the following properties.

- | | |
|--|--|
| $f'(x) < 0$ on $(-\infty, -6) \cup (1, 3)$ | $f''(x) > 0$ on $(-\infty, -6) \cup (3, \infty)$ |
| $f'(x) > 0$ on $(-6, 1) \cup (3, \infty)$ | $f''(x) < 0$ on $(-6, 3)$ |
- DECR CC↑
 INCR



4. Use the second derivative test to find the local extrema of $f(x) = x^3 - 3x$.

$$f' = 3x^2 - 3 = 0 \quad f''(-1) = -6 < 0 \Rightarrow \text{REL MAX AT } x = -1$$

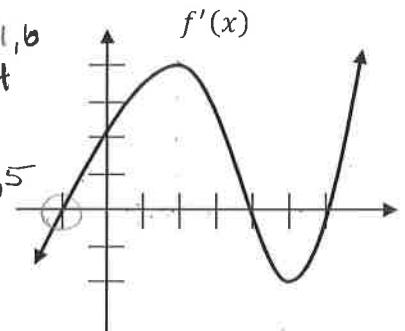
$$x = \pm 1 \quad \Rightarrow \text{REL MAX AT } (-1, 2)$$

$$f'' = 6x \quad f''(1) = 6 > 0 \Rightarrow \text{REL MIN AT } x = 1$$

$$\Rightarrow \text{REL MIN AT } (1, -2)$$

5. Use the graph of $f'(x)$ at the right to answer parts (a) through (i).

- What is the slope of $f(x)$ at $x = 2$? 4
- For which x -values does $f(x)$ have a horizontal tangent line? $x = -1, 4, 6$
- Find the intervals where $f(x)$ is increasing. $[-1, 4] \cup [6, \infty)$
- Find the x -values where $f(x)$ has a relative minimum. $x = -1, 6$
- Find the x -values where $f(x)$ has a relative maximum. $x = 4$
- Is $f(x)$ increasing or decreasing at $x = 5$? DECR
- Is $f(x)$ concave up or concave down at $x = 4$? CC↓
- Find the x -values where $f(x)$ has an inflection point. $x = 2, 5$
- $\lim_{x \rightarrow -\infty} f(x)$. ∞



6. Match each observation 1-5 to each conclusion A-E.

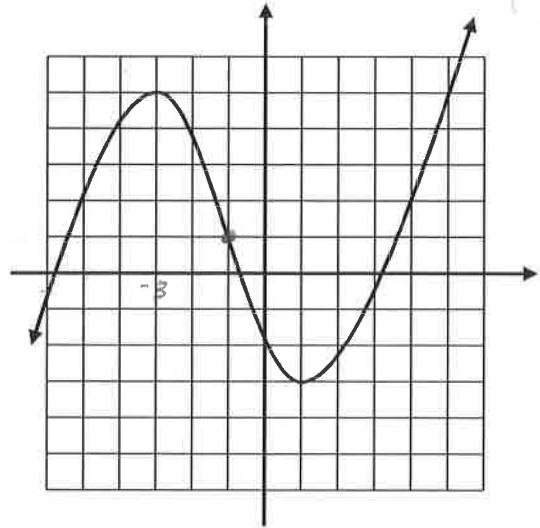
- The point $(3, 4)$ is on the graph of $f'(x)$. C
- The point $(3, 4)$ is on the graph of $f(x)$. D
- The point $(3, 4)$ is on the graph of $f''(x)$. B
- The point $(3, 0)$ is on the graph of $f'(x)$ and $(3, 4)$ is on the graph of $f''(x)$. A
- The point $(3, 0)$ is on the graph of $f'(x)$ and $(3, -4)$ is on the graph of $f''(x)$. E

- $f(x)$ has a relative minimum at $x = 3$.
- At $x = 3$, the graph of $f(x)$ is concave up.
- At $x = 3$, the tangent line to the graph of $f(x)$ has slope 4.
- At $x = 3$, the value of $f(x)$ is 4.
- $f(x)$ has a relative maximum at $x = 3$.

7. In the graph of $f(x)$ at the right, assume that $(-1, 1)$ is an inflection point. Use the graph to answer parts (a) through (k).

- Find the intervals where $f(x)$ is decreasing.
- Find the intervals where $f'(x) > 0$.
- Find all the critical points.
- Find the intervals where $f(x)$ is concave up.
- Find the intervals where $f(x)$ is concave down.
- Is $f'(-4)$ positive, negative or zero?
- Is $f''(-4)$ positive, negative, or zero?
- Is $f'(0)$ positive, negative or zero?
- Is $f''(0)$ positive, negative or zero?
- Is $f'(1)$ positive, negative, or zero?
- Is $f''(1)$ positive, negative, or zero?

$\boxed{(-3, 1)}$
 $(-\infty, -3) \cup (1, \infty)$
 $x = -3, 1$
 $(1, \infty)$
 $(-\infty, -1)$
 POSITIVE
 NEGATIVE
 NEGATIVE
 POSITIVE
 \circ
 POSITIVE



8. Await the instructor directions for the following functions.

- $f(x) = x^3 + 3x^2 - 9x + 4$
- $f(x) = 1 + x^{2/3}$
- $f(x) = \frac{x^2+4}{2x}$
- $f(x) = \arcsin(1/x)$
- $f(x) = x^4 - 4x^3 + 10$
- $f(x) = 6x^{2/3} - 4x$
- $f(x) = \frac{(x+1)^2}{1+x^2}$
- $f(x) = e^{2/x}$

$$(8) \text{ a) } f(x) = x^3 + 3x^2 - 9x + 4$$

$$f' = 3x^2 + 6x - 9 = 0$$

$$f'' = 6x + 6 = 0$$

$$f''(-3) = -12 < 0 \Rightarrow \text{REL MAX}$$

$$3(x^2 + 2x - 3) = 0$$

$$x = -1$$

AT $x = -3$

$$3(x+3)(x-1) = 0$$

$$x = -3, 1$$

$$f'' \leftarrow \begin{array}{c} - \\ + \end{array} \begin{array}{c} + \\ -1 \end{array}$$

$$f''(1) = 12 > 0 \Rightarrow \text{REL MIN}$$

AT $x = 1$

$$f' \leftarrow \begin{array}{c} + \\ -3 \end{array} \begin{array}{c} - \\ 1 \end{array} \begin{array}{c} + \\ \end{array}$$

f INCR $(-\infty, -3] \cup [1, \infty)$

SINCE $f' > 0$ ON THESE INTS.

f CC↑ $(-1, \infty)$ SINCE $f'' > 0$ ON $(-1, \infty)$

f CC↓ $(-\infty, -1)$ SINCE $f'' < 0$ ON $(-\infty, -1)$

POI @ $x = -1 \Rightarrow f''$ CHANGES SIGN @ $x = -1$

$$\text{b) } f(x) = 1 + x^{2/3}$$

$$f'' = -\frac{2}{9}x^{-4/3} < 0 \text{ FOR ALL } x \neq 0$$

$$f' = \frac{2}{3}x^{-1/3}$$

f CC↑ ON $(-\infty, 0) \cup (0, \infty)$

$f'(0)$ DNE

$$\leftarrow \begin{array}{c} -- \\ | \\ ++ \end{array} \rightarrow$$

f' CHANGES FROM - TO + AT $x = 0 \Rightarrow$

REL MIN @ $x = 0$.

f INCR $[0, \infty) \Rightarrow f' > 0$ ON $(0, \infty)$

f DECR $(-\infty, 0] \Rightarrow f' < 0$ ON $(-\infty, 0)$

$$\text{c) } f(x) = \frac{x^2 + 4}{2x}$$

f' CHANGES FROM - TO + AT $x = 2 \Rightarrow$ REL MAX
@ $x = 2$

$$f' = \frac{(2x)(2x) - (x^2 + 4)(2)}{(2x)^2} = 0$$

f' CHANGES FROM - TO + AT $x = 2 \Rightarrow$ REL MIN
@ $x = 2$

$$= \frac{4x^2 - 2x^2 - 8}{4x^2}$$

$$= \frac{2x^2 - 8}{4x^2} = \frac{x^2 - 4}{2x^2}$$

$$= \frac{(x-2)(x+2)}{2x^2}$$

$$f' \leftarrow \begin{array}{c} + \\ - \end{array} \begin{array}{c} \oplus \\ 0 \end{array} \begin{array}{c} - \\ 2 \end{array} \begin{array}{c} + \\ \end{array}$$

$$f'' = \frac{(2x^2)(2x) - (x^2 - 4)(4x)}{4x^4} \quad f'' \text{ one @ } x = 0$$

$$= \frac{4x^3 - 4x^3 + 16x}{4x^4} = \frac{4}{x^3}$$

$$f'' \leftarrow \begin{array}{c} -- \\ \oplus \end{array} \begin{array}{c} ++ \\ \end{array}$$

f INCR $(-\infty, -2] \cup [2, \infty) \Rightarrow f' > 0$

ON $(-\infty, -2) \cup (2, \infty)$

f DECR $[-2, 0) \cup (0, 2] \Rightarrow f' < 0$

ON $(-2, 0) \cup (0, 2)$

f CC↓ ON $(-\infty, 0) \Rightarrow f'' < 0$ ON $(-\infty, 0)$

f CC↑ ON $(0, \infty) \Rightarrow f'' > 0$ ON $(0, \infty)$

NO POI

DOMAIN $(-\infty, -1] \cup [1, \infty)$

d) $f = \sin^{-1}(1/x)$

$$f' = \frac{1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2} = \frac{-1}{x^2 \sqrt{1-1/x^2}} = \frac{-1}{\sqrt{x^4-x^2}} = -\frac{1}{(x^4-x^2)^{1/2}}$$

$$= \frac{-1}{x^2 \sqrt{1-1/x^2}} = 0$$

$$f'' = \frac{1}{2} (x^4-x^2)^{-3/2} (4x^3-2x)$$

f' und AT $x=0$. (not in domain)

f'' und AT $x=\pm 1$



f DECR $(-\infty, -1] \cup [1, \infty)$

$\rightarrow f' < 0$ $(-\infty, -1) \cup (1, \infty)$

$$= \frac{2x^3 - x}{(x^4 - x^2)^{3/2}} = \frac{x(2x^2 - 1)}{(x^4 - x^2)^{3/2}}$$

$f'' = 0$ AT $x = \pm \sqrt{\frac{1}{2}}$ \Rightarrow NOT IN DOMAIN



f CC↓ $(-\infty, -1) \rightarrow f'' < 0$ on $(-\infty, -1)$

f CC↑ $(1, \infty) \rightarrow f'' > 0$ on $(1, \infty)$

e) $f(x) = x^4 - 4x^3 + 10$

$$f' = 4x^3 - 12x^2 = 0$$

$$4x^2(x-3) = 0$$



f INCR $[3, \infty) \rightarrow f' > 0$ $(3, \infty)$

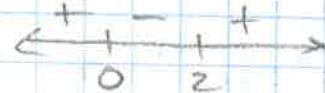
f DECR $(-\infty, 3] \rightarrow f' > 0$ $(-\infty, 3)$

REL MIN @ $x=3 \rightarrow f'$ changes

FROM - TO + AT $x=3$.

$$f'' = 12x^2 - 24x = 0$$

$$12x(x-2) = 0$$



f CC↑ $(-\infty, 0) \cup (2, \infty) \rightarrow f'' > 0$ on

THESE INTERVALS

f CC↓ $(0, 2) \rightarrow f'' < 0$ on $(0, 2)$

P.O.I AT $x=0, 2 \rightarrow f''$ changes sign
AT $x=0, 2$

$$f) f = 6x^{2/3} - 4x$$

$$f' = 4x^{-1/3} - 4 = 0$$

$$4(\frac{1}{\sqrt[3]{x}} - 1) = 0$$

$$\frac{1}{\sqrt[3]{x}} = 1 \quad f'(0) \text{ DNE}$$

$$x = 1$$

$$f'' = -\frac{4}{3}x^{-4/3} = -\frac{4}{3\sqrt[3]{x^4}}$$

f'' DNE @ $x=0$

$f \text{ CC}\uparrow \text{ on } (-\infty, \infty)$
NO POI

$f \text{ INCR } [0, 1] \rightarrow f' > 0 (0, 1)$
 $f \text{ DECR } (-\infty, 0] \cup [1, \infty) \rightarrow$
 $f' < 0 \text{ on } (-\infty, 0) \cup (1, \infty)$

REL MIN @ $x=0 \rightarrow f'$ changes from
From - TO + AT $x=0$

REL MAX @ $x=1 \rightarrow f'$ changes from
+ TO - AT $x=1$.

$$g) f(x) = \frac{(x+1)^2}{1+x^2}$$

$$f' = \frac{(1+x^2)(2(x+1)) - (x+1)^2(2x)}{(1+x^2)^2}$$

$$= \frac{2(x+1)[(1+x^2) - x(x+1)]}{(1+x^2)^2}$$

$$= \frac{2(x+1)(1-x)}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$f' \quad \begin{array}{c} \text{---} \\ -1 \end{array} \quad \begin{array}{c} ++ \\ 1 \end{array} \quad \begin{array}{c} \text{---} \\ 1 \end{array}$$

$f \text{ DECR } (-\infty, -1] \cup [1, \infty) \rightarrow$
 $f' < 0 \text{ on } (-\infty, -1) \cup (1, \infty)$

$f \text{ INCR } [-1, 1] \rightarrow f' > 0$
on $(-1, 1)$

REL MIN @ $x=-1 \rightarrow f'$ changes - TO +
REL MAX @ $x=1 \rightarrow f'$ changes + TO -

$$f'' = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1+x^2) \cdot 2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{-4(1+x^2)[+(1+x^2) + (1-x^2) \cdot 2]}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)(1+x^2+2-2x^2)}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)(3-x^2)}{(1+x^2)^4}$$

$f \text{ CC}\downarrow (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}) \rightarrow f'' < 0$
ON THESE INTERVALS

$f \text{ CC}\uparrow (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty) \rightarrow f'' > 0$

on THESE INTERVALS

POI @ $x = \pm\sqrt{3}, 0 \rightarrow f'' \text{ CHANGES}$

$$h) f(x) = e^{4x}$$

$$f' = e^{4x} \cdot \frac{-2}{x^2} = \frac{-2e^{4x}}{x^2} = 0$$

$$f'(x) \neq 0$$

$f'(x)$ DNE @ $x=0$ (NOT IN
domain of f)

$$f' \leftarrow \overbrace{\quad\quad\quad\quad\quad\rightleftharpoons}$$

$$f \text{ DECR } (-\infty, 0) \cup (0, \infty)$$

$$f'' = \frac{x^2(1+2e^{2x}) + 2e^{2x} \cdot 2x}{x^4}$$

$$= \frac{4e^{2x}(1+x)}{x^4}$$

$$= \frac{4e^{2x}(1+x)}{x^4} = 0 \text{ AT } x = -1$$

$$\begin{array}{c} - \\ \leftarrow \end{array} \begin{array}{c} + \\ \oplus \end{array} \begin{array}{c} + \\ \rightarrow \end{array}$$

$$f \text{ CC } \downarrow (-\infty, -1) \rightarrow f'' < 0 \text{ on } (-\infty, -1)$$

$$f \text{ CC } \uparrow (-1, 0) \cup (0, \infty) \rightarrow f'' > 0 \text{ on }$$

$$(-1, 0) \cup (0, \infty)$$

$$\text{POI } @ x = -1 \rightarrow f'' \text{ changes sign at } x = -1$$